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BOUNDARY VALUE PROBLEM FOR CLASS OF ANALYTIC FUNCTIONS DEFINED IN $\Omega = R \times (0, a)$

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Abstract

In this article, we consider the following boundary value problem for a class of analytic functions defined in a domain $\Omega = R \times (0, a)$, where R is the real line and $a > 0$. The boundary value of the function $f(z) = f(x + iy)$ when $y \to 0+$ is a distribution.

1. Introduction

The boundary value of the considered functions will be the distributions of the following spaces: D' the space of the Schwartz distributions on R, S' the space of tempered distributions and distributions of the space O'_α, $\alpha \in R$ introduced by Bremermann [1].

Let $f(z)$ be a given function defined in the domain Ω, suppose that there are real numbers $M \geq 0$ and s such that

$$|f(z)| = |f(x + iy)| \leq My^{-s}, \quad z = x + iy \in \Omega. \tag{1}$$

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For such a function $f(x+iy)$, we shall prove that converges in distributional sense to the distribution $f(x+i\cdot 0)$ as $y \to 0+$.

First, we will prove the following lemma:

Lemma 1. *Let* $f(x+iy)$ *be analytic function in the domain* Ω *and satisfies condition* (1).

If $0 < y_1 < y_2 < a$, *then holds the relation*

$$\int_R f(x+iy_1)\varphi(x)dx - \int_R f(x+iy_2)\varphi(x)dx = i\iint_G \varphi(x)f(x+iy)dxdy, \qquad (2)$$

for $\varphi \in D$, D *is the space of test functions.*

Proof. Let $\varphi \in D$, then there is an $r > 0$ such that $\operatorname{supp}\varphi \subset [-r, r]$. Now we will apply the Green's theorem to the functions

$$P(x, y) = \varphi(x)f(x+iy) = Q(x, y),$$

on the rectangle $G : [-r, r] \times [y_1, y_2]$ in Ω

$$\int_{\partial G} P(x, y)dx + Q(x, y)dy = \iint_G \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right]dxdy,$$

where ∂G is the boundary of the rectangle G,

$$\frac{\partial Q}{\partial x} = \varphi'(x)f(x+iy) + \varphi(x)f'(x+iy),$$

$$\frac{\partial P}{\partial y} = i\varphi(x)f'(x+iy).$$

$$\int_{\partial G} \varphi(x)f(x+iy)dx = \int_{-r}^{r} \varphi(x)f(x+iy_1)dx - \int_{y_1}^{y_2} \varphi(r)f(r+iy)dy$$

$$- \int_{-r}^{r} \varphi(x)f(x+iy_2)dx - \int_{y_1}^{y_2} \varphi(-r)f(-r+iy)dy.$$

Since $\varphi(r) = \varphi(-r) = 0$, we have that the above integral is equal to

$$\int_{-r}^{r} \varphi(x) f(x + iy_1)\, dx - \int_{-r}^{r} \varphi(x) f(x + iy_2)\, dx.$$

Of course, we may write $\left(\int_{-\infty}^{\infty} \varphi(x) f(x + iy_1)\, dx - \int_{-\infty}^{\infty} \varphi(x) f(x + iy_2)\, dx \right)$.

$$\int_{\partial G} \varphi(x) f(x + iy)\, dy = \int_{-r}^{r} \varphi(x) f(x + iy_1) \cdot 0 + \int_{y_1}^{y_2} \varphi(r) f(r + iy)\, dy$$

$$- \int_{-r}^{r} \varphi(x) f(x + iy_2) \cdot 0 - \int_{y_1}^{y_2} \varphi(-r) f(-r + iy)\, dy = 0.$$

Replacing in the Green's formula, we obtain

$$\int f(x + iy_1) \varphi(x)\, dx - \int f(x + iy_2) \varphi(x)\, dx$$

$$= \iint_G [\varphi'(x) f(x + iy) + \varphi(x) f'(x + iy) - i\varphi(x) f'(x + iy)]\, dx dy. \tag{3}$$

Let us now consider the integral

$$\iint_G \varphi'(x) f(x + iy)\, dx dy = \int_{y_1}^{y_2} dy \int_{-r}^{r} \varphi'(x) f(x + iy)\, dx.$$

After partial integration in the integral

$$\int_{-r}^{r} \varphi'(x) f(x + iy)\, dx,$$

we give the integral

$$- \iint_G \varphi(x) f'(x + iy)\, dx dy,$$

if we replace in (3), we obtain

$$\int f(x+iy_1)\varphi(x)dx - \int f(x+iy_2)\varphi(x)dx$$

$$= \iint_G [-\varphi(x)f'(x+iy) + \varphi(x)f'(x+iy) - i\varphi(x)f'(x+iy)]dxdy$$

$$= -i\iint_G \varphi(x)f'(x+iy)dxdy = i\iint_G \varphi'(x)f(x+iy)dxdy.$$

Thus, we can write

$$\int_{-\infty}^{\infty} f(x+iy_1)\varphi(x)dx - \int_{-\infty}^{\infty} f(x+iy_2)\varphi(x)dx = i\int_{-\infty}^{\infty}\int_{y_1}^{y_2} \varphi'(x)f(x+iy)dxdy.$$

The proof is done.

In the sequel, we shall prove the following theorem:

Theorem 1. *Let* $f(x+iy)$ *is analytic function in the domain* Ω *and satisfies condition* (1), *then*

$$\lim_{y\to 0+}\int f(x+iy)\varphi(x)dx = < f(x+i\cdot 0),\ \varphi(x) >, \textit{exists for } \varphi \in D,$$

and $f(x+i\cdot 0)$ *is a distribution of* D'.

Proof. Since $|f(x+iy)| \le My^{-s}$ thus for any fixed $y > 0$, the function $f(x+iy)$ relative x is a regular distribution, and consequently, in view [9], it is sufficient to prove the existence of the limit

$$\lim_{y\to 0+}\int f(x+iy)\varphi(x)dx, \text{ for } \varphi \in D.$$

Let $0 < y_1 < y_2 < a$, then by Lemma 1, we have

$$\int f(x+iy_1)\varphi(x)dx - \int f(x+iy_2)\varphi(x)dx = i\int_{-\infty}^{\infty}\int_{y_1}^{y_2} \varphi'(x)f(x+iy)dxdy.$$

Now

$$\iint \varphi'(x)f(x+iy)dxdy = \int_{-\infty}^{\infty} \varphi'(x)dx \int_{y_1}^{y_2} f(x+iy)dy,$$

$$\left|\int_{y_1}^{y_2} f(x+iy)dy\right| \le \int_{y_1}^{y_2} |f(x+iy)|dy \le M\int_{y_1}^{y_2} \frac{dy}{y^s}.$$

Case (i) If $s < 1$, then the function $\frac{1}{y^s}$ is locally integrable and thus as $y_1, y_2 \to 0+$, then the integral $\int_{y_1}^{y_2} y^{-s}dy$ tends to zero, consequently, we have

$$\left|\int f(x+iy_1)\varphi(x)dx - \int f(x+iy_2)\varphi(x)dx\right| \le \int |\varphi'(x)|dx \int_{y_1}^{y_2} y^{-s}dy.$$

Since $\int |\varphi'(x)|dx < \infty$, by using the Cauchy criterion, we conclude that

$$\lim_{y\to 0+} \int f(x+iy)\varphi(x)dx$$

exists.

From this, we have that

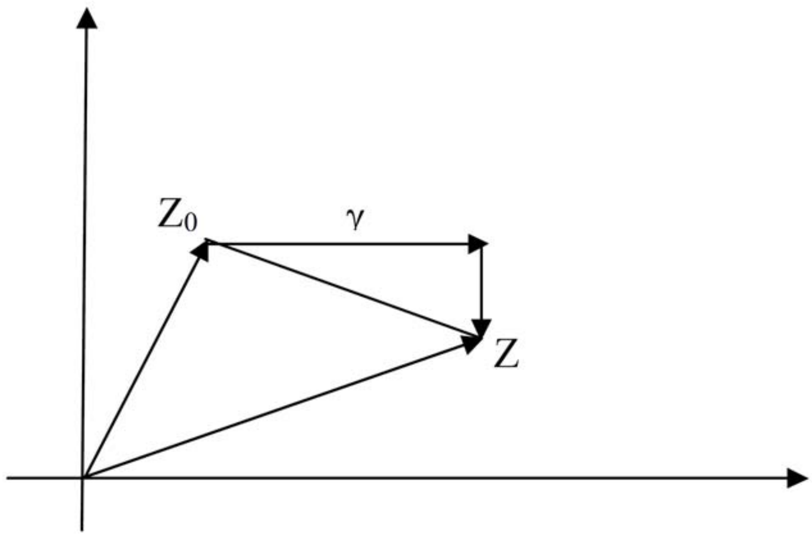
$$f(x+iy) \to f(x+i0) \text{ as } y \to 0+ \text{ in } D'$$

sense.

Case (ii) Let $s = 1$. Consider first the function

$$f_1(z) = \int_{z_0}^{z} f(\zeta)d\zeta = \int_{\gamma} f(\zeta)d\zeta,$$

where γ is a suitable path, see the figure



$$z_0 = x_0 + iy_0 \text{ and } z = x + iy.$$

For our purpose, we may take that $y \leq y_0$ since $y \to 0+$.

Now

$$f_1(z) = \int_{x_0}^{x} f(t + iy_0)dt - i\int_{y}^{y_0} f(x + it)dt.$$

Since

$$|f_1(z)| \leq \int_{x_0}^{x} \frac{1}{y_0} dt + \int_{y}^{y_0} \frac{1}{t} dt = \frac{x - x_0}{y_0} + \log y_0 - \log y,$$

we have

$$\int f(x + iy_1)\varphi(x)dx - \int f(x + iy_2)\varphi(x)dx$$

$$= i\int\int \varphi'(x)f(x + iy)dxdy$$

$$= i\int\int \varphi'(x)f_1'(x + iy)dxdy$$

$$= i\int dy\left[f_1(x+iy)\varphi'(x)\Big|_{-r}^{r} - \int_{-r}^{r}\varphi''(x)f_1(x+iy)dx\right]$$

$$= i\int dy\left[-\int\varphi''(x)f_1(x+iy)dx\right]$$

$$= -i\int\varphi''(x)dx\int_{y_1}^{y_2}f_1(x+iy)dy.$$

$$\left|\int f(x+iy_1)\varphi(x)dx - \int f(x+iy_2)\varphi(x)dx\right|$$

$$= \left|\int\varphi''(x)dx\int_{y_1}^{y_2}f_1(x+iy)dy\right|$$

$$\leq \int|\varphi''(x)|dx\int_{y_1}^{y_2}\left[\frac{x-x_0}{y_0} + \log y_0 - \log y\right]dy.$$

We may assume that $y \leq \min(a, 1)$ and we have

$$\int|\varphi''(x)|\cdot|x-x_0|dx\left(y_2 - y_1 + \log y_0\cdot(y_2-y_1) - \int_{y_1}^{y_2}\log y dy\right).$$

Since $\int_{-r}^{r}|x-x_0|\cdot|\varphi''(x)| < \infty$ and $\log y$ is locally integrable, we conclude as in the Case (i) that exists

$$\lim_{y\to 0+}\int f(x+iy)\varphi(x)dx = \langle f(x+i\cdot 0), \varphi(x)\rangle, \text{ for } \varphi\in D.$$

For $s > 1$, by induction, one can verify the existence of the limit

$$\lim_{y\to 0+}\int f(x+iy)\varphi(x)dx = \langle f(x+i\cdot 0), \varphi(x)\rangle, \text{ for } \varphi\in D.$$

So, we proved for real s that $f(x+iy)\to f(x+i0)$ in D' as $y\to 0+$.

2. Let now assume that $\varphi \in S$. Also, we will prove that the relation

$$\int f(x+iy_1)\varphi(x)dx - \int f(x+iy_2)\varphi(x)dx = i\int_{-\infty}^{\infty}\int_{y_1}^{y_2}\varphi'(x)f(x+iy)dxdy$$

holds.

Proof. The proof is similar. Let G be the rectangle $[-r, r]\times[y_1, y_2]$ in Ω. Then we apply the Green's theorem to the functions $P(x, y) = f(x+iy)\varphi(x) = Q(x, y)$ on G. Then in the same way as in the case of D, we obtain the following relation:

$$\int_{-r}^{r} f(x+iy_1)\varphi(x)dx - \int_{-r}^{r} f(x+iy_2)\varphi(x)dx$$

$$= i\int_{-r}^{r}\int_{y_1}^{y_2}\varphi'(x)f(x+iy)dxdy + \varphi(r)\int_{y_1}^{y_2} f(r+iy)dy - \varphi(-r)\int_{y_1}^{y_2} f(-r+iy)dy.$$

Now

$$\left|\varphi(r)\int_{y_1}^{y_2} f(r+iy)dy\right| \le |\varphi(r)|\int_{y_1}^{y_2}\frac{dy}{y^s} \text{ and } \left|\varphi(-r)\int_{y_1}^{y_2} f(-r+iy)dy\right| \le |\varphi(-r)|\int_{y_1}^{y_2}\frac{dy}{y^s}.$$

If $s < 1$, then $\int_{y_1}^{y_2}\frac{dy}{y^s} \to 0$ as $y_1, y_2 \to 0+$, also if $r \to \infty$, then $|\varphi(r)|$ and $|\varphi(-r)| \to 0$. So, we can first assume that $r \to \infty$ and we give

$$\int f(x+iy_1)\varphi(x)dx - \int f(x+iy_2)\varphi(x)dx = i\int_{-\infty}^{\infty}\int_{y_1}^{y_2}\varphi'(x)f(x+iy)dxdy.$$

Finally, since

$$\left|\int_{-\infty}^{\infty}\varphi'(x)dx\int_{y_1}^{y_2} f(x+iy)dy\right| \le \int_{-\infty}^{\infty}|\varphi'(x)|dx\cdot\int_{y_1}^{y_2}\frac{dy}{y^s},$$

and since

$$\int_{-\infty}^{\infty} |\varphi'(x)| dx < \infty,$$

we conclude, as in the case of D, that

$$\lim_{y \to 0+} \int f(x + iy)\varphi(x) dx =< f(x + i \cdot 0), \varphi(x) >, \text{ for } \varphi \in S.$$

Further, since $|f(x + iy)| \leq \frac{M}{y}$, we have that $f(x + iy) \in S'$, and from the completeness of S', we conclude that $f(x + iy) \to f(x + i \cdot 0)$ in S' as $y \to 0+$. Similarly follows for every $s \in R$.

3. In this section, we consider the space O_α, $\alpha \in R$ and refer to the following theorem:

Theorem 2. *Let* $f(z)$ *be analytic in the open upper half plane* Λ^+ *with* $f(z) = O\left(\frac{1}{|z|}\right)$ *as* $|z| \to \infty$ *in* Λ^+. *If* $f(x + iy)$ *converge to the boundary value* $f(x + i \cdot 0)$ *in* D' *as* $y \to 0+$, *then*

(i) $f(x + i \cdot 0) \in O'_\alpha$ *for all* $\alpha < 0$; *and*

(ii) $f(x + iy)$ *converge to* $f(x + i \cdot 0)$ *in the* O'_α *topology as* $y \to 0+$, $\alpha < 0$. *Additionally, if* $-1 \leq \alpha < 0$, *we have that*

(iii) $$\frac{1}{2\pi i} < f(x + i \cdot 0), \frac{1}{t - z} >= \begin{cases} f(x + iy), & y > 0, \\ 0, & y < 0. \end{cases}$$

This important result in the boundary values problems in the theory of distributions is given in [2].

Here, we assume that the function $f(z)$ is analytic in the open upper half plane Λ^+ with the following properties:

(a) $f(z) = O\left(\frac{1}{|z|}\right)$ as $|z| \to \infty$ in Λ^+.

(b) $|f(x + iy)| \leq My^{-s}$ in some strip $R \times (0, a)$, $a > 0$. Then, we state the following theorem:

Theorem 3. *Let* $f(z)$ *be analytic in* Λ^+ *with properties* (a) *and* (b). *Then as* $y \to 0+$, *the function* $f(x + iy)$

(i) *converges in* D' *to the distribution* $f(x + i \cdot 0)$;

(ii) *converges to* $f(x + i \cdot 0)$ *in* O'_α *topology, for* $\alpha < 0$. *Additionally, if* $-1 \leq \alpha < 0$, *we have that*

(iii) $$\frac{1}{2\pi i} < f(t + i \cdot 0), \frac{1}{t - z} >= \begin{cases} f(x + iy), & y > 0, \\ 0, & y < 0. \end{cases}$$

The function $\frac{1}{2\pi i(t - z)}$ *for* Im $z \neq 0$ *is the Cauchy kernel.*

Proof. The proof immediately follows from Theorem 1 and Lemma 2.

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